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Weak Asymptotics for the Generating Polynomials of the Stirling Numbers of the Second Kind

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For the horizontal generating functions $P_n(z)=\sum_{k=1}^{n} S(n,k)\,z^k$ of the Stirling numbers of the second kind, a weak asymptotic is established, as $n\to\infty$. The distribution function F_n of the zeros of $Q_n(z)=P_n(nz)$ is investigated and by using the Stieltjes transformation, the limit of F_n in the sense of weak convergence is deduced.

Key Words: Stirling numbers of the second kind; generating functions; asymptotic zero distribution.

1. INTRODUCTION AND SUMMARY

In this paper we deduce the asymptotic distribution function of the zeros for the horizontal generating function of the Stirling numbers of the second kind $S(n,k)$, which are defined by the following double generating function (see [2, p. 50]):

$$\exp\{z(e^u-1)\}=:1+\sum_{1\leqslant k\leqslant n<\infty} S(n,k)\,\frac{u^n}{n!}\,z^k, \qquad z,u\in\mathbb{C}. \tag{1.1}$$

The horizontal generating functions $P_n(z)$ are the coefficients of the following power series:

$$\exp\{z(e^u-1)\}=:1+\sum_{n=1}^{\infty}\frac{P_n(z)}{n!}\,u^n, \qquad z,u\in\mathbb{C}.$$

Concerning the zeros of P_n there is one result: The zeros are simple, real, and not greater than 0 (see [2, p. 271]). In [5] two asymptotic expansions for $Q_n(z):=P_n(nz)$ are deduced by means of the saddle point method. With $\Phi\colon \mathbb{C}\backslash\{-e,0\}\to\mathscr{A}\cup\Gamma_+$, which maps $\mathbb{C}\backslash[-e,0]$ conformally on

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$\mathscr{A} := \{w \in \mathbb{C}\backslash\{0\} : w > -1$ or $w = a + ib,\ a > -b \cot b,\ b \in (-\pi, \pi)\backslash\{0\}\}$, and $(-e, 0)$ one-one on $\Gamma_+ := \{w \in \partial\mathscr{A} : \Im(w) > 0\}$, the following results are obtained:

(i) With $\phi \in (0, \pi)$ there is the oscillating asymptotics

$$Q_n(x(\phi)) = k_n(\phi)\left(\sin\left(n\left(\pi - \phi + \frac{\sin^2\phi}{\phi}\right) + \eta(\phi)\right) + \mathcal{O}\left(\frac{1}{n}\right)\right)$$

with $k_n(\phi) > 0$, $\eta(\phi)$ bounded by $\frac{\pi}{2}$ and π, $x(\phi) \in (-e, 0)$, $x(\phi)\, w(\phi)\, e^{w(\phi)} = 1$,

$$x(\phi) = -\frac{\sin\phi}{\phi}\, e^{\phi\cot\phi} \quad \text{and} \quad \Phi(x(\phi)) = w(\phi) := \frac{\phi}{\sin\phi}\, e^{i(\pi - \phi)}. \tag{1.2}$$

(ii) With $z \in \mathbb{C}\backslash[-e, 0]$, $w = \Phi(z)$ and $zwe^w = 1$, it holds that

$$Q_n(z) = \frac{n!}{\sqrt{2\pi n}}\, \frac{1}{w^n} \exp\left\{\frac{n}{w}(1 - e^{-w})\right\}(1 + w)^{-1/2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

where the $\mathcal{O}$-term holds uniformly on every compact subset of $\mathbb{C}\backslash[-e, 0]$. Examining these results, we can presume that nearly all zeros of Q_n are in $[-e, 0]$, because for sufficiently large n the asymptotic in (ii) is not equal to 0. Therefore, we discuss the position of the zeros by investigating the distribution function of the zeros for $Q_n(z)$:

$$F_n(\xi) := \frac{1}{n} N_n(\xi), \quad N_n(\xi) := |\{x \leqslant \xi : Q_n(x) = 0\}|, \qquad n \in \mathbb{N}.$$

By using the Stieltjes transformation, we get a weak asymptotic; i.e., we prove the weak convergence of the sequence F_n to a distribution function F of the following type,

$$F(x) = \begin{cases} 0, & x \leqslant -e \\ 1 + \dfrac{1}{\pi}\left(\Im\left(\dfrac{1}{w}\right) - \mathrm{ph}(w)\right), & x \in (-e, 0) \\ 1, & x \geqslant 0, \end{cases}$$

with $\mathrm{ph}(w) \in (0, \pi)$, $w \in \partial\mathscr{A}$, $xwe^w = 1$. With the above parametrization (1.2), it holds further that

$$F\left(-\frac{\sin\phi}{\phi}\, e^{\phi\cot\phi}\right) = \frac{1}{\pi}\left(\phi - \frac{\sin^2\phi}{\phi}\right), \qquad \phi \in (0, \pi).$$

2. WEAK ASYMPTOTICS

To investigate the position of the zeros of Q_n, which are all simple, real, and not greater than 0 (see [2, p. 271]), we define

$$Q_n(z) = n^n \prod_{\nu=1}^{n} (z - x_{n,\nu})$$

$$\text{with} \quad -\infty < x_{n,1} < \cdots < x_{n,n} = 0, \quad n \in \mathbb{N},$$

$$N_n(\xi) := |\{\nu \in \{1, ..., n\} : x_{n,\nu} \leqslant \xi\}|, \qquad n \in \mathbb{N}, \qquad \text{and}$$

$$F_n(\xi) := \frac{1}{n} N_n(\xi), \qquad n \in \mathbb{N}.$$

For computing the limit probability distribution F in the sense of weak convergence, we observe the logarithmic derivative

$$h_{F_n}(z) := \frac{1}{n} \frac{Q_n'(z)}{Q_n(z)} = \frac{1}{n} \sum_{\nu=1}^{n} \frac{1}{z - x_{n,\nu}}$$

$$= \int_{-\infty}^{0} \frac{1}{z-t}\, dF_n(t), \qquad z \in \mathbb{C} \backslash (-\infty, 0], \qquad (2$$

which represents the Stieltjes transform of F_n. The following theorem supplies the limit of this sequence.

THEOREM 2.1. *With the above mentioned notations, h_{F_n} and F_n hold:*

(i) *$h_{F_n}(z)$ converges to $h(z) := e^{\Phi(z)} - 1$ (compactly) on $\mathbb{C} \backslash [-e, 0]$.*

(ii) *The sequence of distributions F_n converges weakly to a distribution F; i.e., if F is continuous at $\xi \in \mathbb{R}$, then $\lim_{n \to \infty} F_n(\xi) = F(\xi)$ and:*

$$\int_{-\infty}^{0} \frac{1}{z-t}\, dF(t) = \int_{-e}^{0} \frac{1}{z-t}\, dF(t) = h(z), \qquad z \in \mathbb{C} \backslash [-e, 0].$$

Proof. (i) Let $K \subset \mathbb{C} \backslash [-e, 0]$ be compact, $z \in K$, and

$$G_n(z) := \frac{n!}{\sqrt{2\pi n}} \frac{1}{\Phi(z)^n} \exp\left\{ n \left(\frac{1}{\Phi(z)} - z \right) \right\} (1 + \Phi(z))^{-1/2}.$$

Because Q_n/G_n converges uniformly to 1 on K (see [5, Theorem 3.3]), it follows from (2.1) that

$$\begin{aligned}\lim_{n\to\infty} h_{F_n}(z) &= \lim_{n\to\infty} \frac{1}{n}\frac{G_n'(z)}{G_n(z)}\\ &= \lim_{n\to\infty} \frac{1}{n}\left(-n\frac{\Phi'(z)}{\Phi(z)} - n\left(\frac{\Phi'(z)}{\Phi(z)^2}+1\right)\right.\\ &\quad \left. -\frac{1}{2}(1+\Phi(z))^{-1}\Phi'(z)\right)\\ &= -\frac{\Phi'(z)}{\Phi(z)} - \frac{\Phi'(z)}{\Phi(z)^2} - 1.\end{aligned}$$

The uniform convergence follows from the compact convergence in [5, Theorem 3.3] and because Φ and Φ' are bounded on K. Let Ψ be the inverse function of Φ, $\Psi(w) = (we^w)^{-1}$, $w \in \mathscr{A}$ (see [5, Lemma 2.3]): then we obtain that $\Phi'(z) = (\Psi'(\Phi(z)))^{-1}$, $\Psi'(w) = e^{-w}(-w^{-2} - w^{-1})$, and further that

$$\begin{aligned}\lim_{n\to\infty} h_{F_n}(z) &= \frac{e^{\phi(z)}}{-\Phi(z)^{-2} - \Phi(z)^{-1}}\left(-\frac{1}{\Phi(z)} - \frac{1}{\Phi(z)^2}\right) - 1\\ &= \frac{e^{\phi(z)}}{1+\Phi(z)}(\Phi(z)+1) - 1 = e^{\phi(z)} - 1 = h(z). \quad \blacksquare\end{aligned}$$

(ii) By using the theorem of Grommer and Hamburger (see [10, p. 175]) and Lemma 1 in [6, p. 383], the proof is completed if $\Re(iyh(iy)) \to 1$ for $0 < y \to \infty$. This follows from [5], Lemma 2.3:

$$\lim_{z\to\infty} zh(z) = \lim_{w\to 0} \Psi(w)\, h(\Psi(w)) = \lim_{w\to 0} \frac{1}{w} e^{-w}(e^w - 1) = 1.$$

Since h is analytic on $\mathbb{C}\backslash[-e, 0]$, the lower boundary in the integral representation $-\infty$ can be replaced by $-e$. $\blacksquare$

Now we focus on determining the distribution F. To use some well-known results concerning the inversion of Stieltjes transforms, we make the following assumption:

$$F \in C_1(-e, 0),\ F'(t) = f(t),\ t \in (-e, 0). \tag{2.2}$$

Thereby, we can apply an inversion formula for the Stieltjes transform

$$h(z) = e^{\Phi(z)} - 1 = \int_{-e}^{0} \frac{f(t)}{z-t}\, dt, \qquad z \in \mathbb{C}\backslash[-e, 0]. \tag{2.3}$$

THEOREM 2.2 *The function f with the Stieltjes transform h and the distribution F hold:*[1]

$$\text{(i)}\quad f(t)=\frac{1}{\pi}\,\Im(\exp(\Phi(t))),\qquad t\in(-e,0),$$

$$\text{(ii)}\quad F(x)=\begin{cases}0, & x\leqslant -e\\ 1+\dfrac{1}{\pi}\left(\Im\left(\dfrac{1}{\Phi(x)}\right)-\mathrm{ph}(\Phi(x))\right), & x\in(-e,0),\ \mathrm{ph}(\Phi(x))\in(0,\pi)\\ 1, & x\geqslant 0,\end{cases}$$

$$\text{(iii)}\quad F(x(\phi))=\frac{1}{\pi}\left(\phi-\frac{\sin^2\phi}{\phi}\right),$$

$$x(\phi)=-\frac{\sin\phi}{\phi}\,e^{\phi\cot\phi},\qquad \phi\in(0,\pi).$$

Proof. (i) Let t be in $(-e,0)$; then by using Theorem 14.1 in [11, p. 126]:

$$f(t)=\lim_{y\to 0_+}\frac{h(t-iy)-h(t+iy)}{2\pi i}. \tag{2.4}$$

To compute $h(t-i0)$ let $z\notin\mathbb{R}$, $\Re(z)\in(-e,0)$, $zwe^w=1$, $w=x+iy$, $y\neq 0$, and $w\in\mathscr{A}$; then we have:

$$\Im(z)=\Im\left(\frac{1}{x+iy}\,e^{-x-iy}\right)=\Im\left(\frac{x-iy}{x^2+y^2}\,e^{-x}e^{-iy}\right)$$

$$=\frac{e^{-x}}{x^2+y^2}(-x\sin y-y\cos y)$$

$$=\frac{e^{-x}}{x^2+y^2}\sin y(-x-y\cot y).$$

So by the definition of $\mathscr{A}$ we can conclude that $\Im(z)$ is less than or greater than 0 if and only if y is greater than or less than 0, respectively. Thus (2.4) becomes:

$$f(t)=\frac{1}{2\pi i}(\exp(\Phi(t))-\exp(\overline{\Phi(t)}))=\frac{1}{\pi}\,\Im(\exp(\Phi(t))).$$

[1] The result in (iii) is not surprising; compare with the sin-term in (1.2).

Because f has been determined by the help of (2.2) we still have to verify that f complies with (2.3). That can be easily considered by the substitutions $t=\Psi(w)$, $w\in\Gamma_+$, and $t=\Psi(w)$, $w\in\Gamma_-$, and the use of the residue theorem.

(ii) By Theorem 2.1 and (2.2) there is a c in $[0, 1)$ satisfying:

$$F(x)=\begin{cases} 0, & x<-e \\ c, & x=-e \\ c+\int_{-e}^{x} f(t)\,dt, & x\in(-e,0) \\ 1, & x\geqslant 0. \end{cases} \tag{2.5}$$

With x in $(-e, 0)$ and $t=\Psi(w)=(we^w)^{-1}$, it follows that

$$\begin{aligned}\int_{-e}^{x}\frac{1}{\pi}\,\Im(\exp(\Phi(t)))\,dt &=\frac{1}{\pi}\,\Im\left(\int_{-1,\,w\in\Gamma_+}^{\Phi(x)} e^w e^{-w}\left(-\frac{1}{w^2}-\frac{1}{w}\right)dw\right)\\ &=\frac{1}{\pi}\,\Im\left(\frac{1}{\Phi(x)}+1-\ln(\Phi(x))+\ln(-1)\right)\\ &=1+\frac{1}{\pi}\left(\Im\left(\frac{1}{\Phi(x)}\right)-\mathrm{ph}(\Phi(x))\right),\end{aligned}$$

which has the limit 0 for $x\to -e_+$ and 1 for $x\to 0_-$. Hence, c must be equal to 0 and the proof is completed.

(iii) Let Φ be in $(0, \pi)$; then due to (ii) and (1.2) we may write:

$$\begin{aligned}F(x(\phi))&=1+\frac{1}{\pi}\left(\Im\left(\frac{1}{\Phi(x(\phi))}\right)-\mathrm{ph}(\Phi(x(\phi)))\right)\\ &=1+\frac{1}{\pi}\left(\Im\left(\frac{\sin\phi}{\phi}\,e^{-i(\pi-\phi)}\right)-\mathrm{ph}\left(\frac{\phi}{\sin\phi}\,e^{i(\pi-\phi)}\right)\right)\\ &=1-\frac{1}{\pi}\,\frac{\sin^2\phi}{\phi}-1+\frac{\phi}{\pi}=\frac{1}{\pi}\left(\phi-\frac{\sin^2\phi}{\phi}\right).\quad\blacksquare\end{aligned}$$

Besides the access to the density f with the help of the Stieltjes transform, there is also the possibility of using an inversion formula for the Mellin transform $g(s)$ of $f_*(t):=f(-t)$:

$$g(s):=\int_0^e f_*(t)\,t^{s-1}\,dt=\int_{-e}^{0} f(t)(-t)^{s-1}\,dt, \qquad \Re(s)>1. \tag{2.6}$$

Indeed, that does not lead to the above mentioned form of f, but we can obtain some interesting results by using this method.

THEOREM 2.3. *Let x be in $(-e, 0)$, $\Re(s) > 1$ and $\ln(s-1) = \ln|s-1| + i\,\mathrm{ph}(s-1)$, $\mathrm{ph}(s-1) \in (-\pi, \pi)$, then*:

(i) $$g(s) = \frac{(s-1)^{(s-1)}}{\Gamma(s+1)},$$

(ii) $$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(s-1)^{s-1}}{\Gamma(s+1)} (-x)^{-s}\, ds, \qquad c > 1,$$

(iii) $$f(x) = \frac{1}{-x\pi^2} \int_0^{\infty} \frac{(-x)^r}{r^r} \sin^2(\pi r) \frac{\Gamma(r)}{1-r}\, dr.$$

Proof. (i) To compute $g(s)$ we observe that for $|z| > e$:

$$e^{\Phi(z)} - 1 = \int_{-e}^{0} \frac{f(t)}{z-t}\, dt = \frac{1}{z} \int_{-e}^{0} f(t) \sum_{k=0}^{\infty} \left(\frac{t}{z}\right)^k dt$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{z^{k+1}} \int_0^e f_*(t)\, t^k\, dt.$$

On the other hand we can develop $e^{\phi(z)} - 1$ into a Bürmann Lagrange series (see [8, pp. 124–125]):

$$e^{\Phi(z)} - 1 = \sum_{k=1}^{\infty} \frac{1}{z^k} \frac{1}{k!} \left[\frac{d^{k-1} e^x (e^{-x})^k}{dx^{k-1}} \right]_{x=0} = \sum_{k=0}^{\infty} \frac{(-1)^k k^k}{(k+1)!} \frac{1}{z^{k+1}}.$$

By the identity theorem for Laurent series, an identity theorem for analytic functions (Theorem 5.81 in [9, p. 186]), and by analytic continuation the claimed form of $g(s)$ is proved.

(ii) By using an inversion formula for $g(s)$ (see [4, p. 409]) it follows that

$$f(x) = f_*(-x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(s-1)^{s-1}}{\Gamma(s+1)} (-x)^{-s}\, ds, \qquad c > 1. \tag{2.7}$$

(iii) To prove (iii) we look at a modified path of integration (Fig. 1) with $\tilde{c} = c - 1$, $d > 0$, $R := |\tilde{c} - id|$, $\alpha_- := \mathrm{ph}(\tilde{c} - id) \in (-\frac{\pi}{2}, 0)$, $\alpha_+ = \mathrm{ph}(\tilde{c} + id) = -\alpha_-$, and:

$$\gamma_1 := \{s \in \mathbb{C} : s = \tilde{c} + iy,\ y \in [-d, d]\},$$
$$\gamma_2 := \{s \in \mathbb{C} : s = Re^{-i\psi},\ \psi \in [\alpha_+, \pi]\},$$
$$\gamma_3 := \{s \in \mathbb{C} : s = re^{-i\pi},\ s \in [-R, -1/R]\},$$
$$\gamma_4 := \{s \in \mathbb{C} : s = R^{-1} e^{i\psi},\ \psi \in [-\pi, \pi]\},$$
$$\gamma_5 := \{s \in \mathbb{C} : s = re^{i\pi},\ r \in [1/R, R]\},$$
$$\gamma_6 := \{s \in \mathbb{C} : s = Re^{i(\pi-\psi)},\ \psi \in [0, \pi - \alpha_+]\}.$$

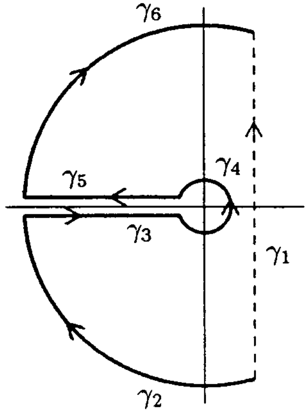

FIG. 1. The path of integration for Theorem 2.3(iii).

By Cauchy's theorem and analytic continuation of g it follows that

$$\int_{c-i\infty}^{c+i\infty} g(s)(-x)^{-s}\,ds$$

$$= \lim_{d\to\infty} \int_{\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6} g(s+1)(-x)^{-s-1}\,ds. \tag{2.8}$$

For investigating $\int_{\gamma_2}$ let $s = Re^{-i\psi}$, $R > 1$, $\psi \in [\alpha_+, \pi]$; then by the functional equation of the Gamma function (see [1, p. 48]) and Stirling's formula (see [7, p. 294]) it holds that

$$\begin{aligned}
g(s+1)(-x)^{-s-1}
&= \frac{s^s}{\Gamma(s+2)}(-x)^{-s-1}\\
&= \frac{s^s}{s+1}\,\frac{\sin(-\pi s)}{\pi}\,\Gamma(-s)(-x)^{-s-1}\\
&= \frac{s^s}{s+1}\,\frac{\sin(-\pi s)}{\pi}\,e^s(-s)^{-s}\sqrt{\frac{2\pi}{-s}}\,(-x)^{-s-1}\left(1+\mathcal{O}\left(\frac{1}{-s}\right)\right)\\
&= \frac{e^s\sqrt{2}}{(-x)^{s+1}\,(s+1)\sqrt{R\pi}}\,\frac{e^{-i\pi s}-e^{i\pi s}}{2i}\\
&\quad \times e^{-(1/2)\,i(-\psi+\pi)}e^{s\ln(R)-is\psi}e^{-s\ln(R)-is(-\psi+\pi)}\left(1+\mathcal{O}\left(\frac{1}{-s}\right)\right)\\
&= \left(\frac{e}{-x}\right)^s \frac{-1}{-x(s+1)\sqrt{2\pi s}}\,(e^{-2i\pi s}-1)\left(1+\mathcal{O}\left(\frac{1}{-s}\right)\right), \qquad d\to\infty.
\end{aligned}$$

Since $\Im(s) \leqslant 0$ and $\Re(s) \leqslant \tilde{c}$, $\int_{\gamma_2}$ is satisfying:

$$\left| \int_{\gamma_2} g(s+1)(-x)^{-s-1}\, ds \right| \leqslant \left(\frac{e}{-x} \right)^{\tilde{c}} \frac{1}{|x|\,(R-1)\,\sqrt{2\pi R}}\, 2\pi R \left(1 + \mathcal{O}\left(\frac{1}{R} \right) \right) \to 0, \quad d \to \infty. \tag{2.9}$$

By analogous argumentation it follows that

$$\left| \int_{\gamma_6} g(s+1)(-x)^{-s-1}\, ds \right| \to 0, \qquad d \to \infty. \tag{2.10}$$

To estimate $\int_{\gamma_4}$ let $s = R^{-1} e^{i\psi}$, $\psi \in [-\pi, \pi]$. Since $(\Gamma(s+2))^{-1} \to 1$ and $|s^s| \to 1$ as $d \to \infty$, it follows that

$$\left| \int_{\gamma_4} g(s+1)(-x)^{-s-1}\, ds \right| \leqslant \max_{s \in \gamma_4} \left| \frac{s^s(-x)^{-s-1}}{\Gamma(s+2)} \right| \frac{2\pi}{R} \to 0, \tag{2.11}$$

as $d \to \infty$. Altogether, we conclude from (2.7), (2.8), (2.9), (2.10), and (2.11) with the substitutions $s = re^{-i\pi}$ and $s = re^{i\pi}$:

$$\begin{aligned} f(x) &= \frac{1}{2\pi i} \left(\int_{\infty}^{0} g(re^{-i\pi}+1)(-x)^{-re^{-i\pi}-1}\, e^{-i\pi}\, dr \right. \\ &\quad \left. + \int_0^{\infty} g(re^{i\pi}+1)(-x)^{-re^{i\pi}-1}\, e^{i\pi}\, dr \right) \\ &= \frac{1}{-2\pi i x} \int_0^{\infty} \frac{(-x)^r}{\Gamma(2-r)} \left((re^{-i\pi})^{re^{-i\pi}} - (re^{i\pi})^{re^{i\pi}} \right) dr \\ &= \frac{1}{-2\pi i x} \int_0^{\infty} \frac{(-x)^r}{\Gamma(2-r)} \left(e^{-r\ln r + ir\pi} - e^{-r\ln r - ir\pi} \right) dr \\ &= \frac{1}{-\pi x} \int_0^{\infty} \frac{(-x)^r}{(1-r)\,\Gamma(1-r)} \frac{\sin(r\pi)}{r^r}\, dr \\ &= \frac{1}{\pi(-x)} \int_0^{\infty} \frac{(-x)^r}{1-r} \frac{\Gamma(r)\sin(r\pi)}{\pi} \frac{\sin(r\pi)}{r^r}\, dr \\ &= \frac{1}{\pi^2(-x)} \int_0^{\infty} \frac{(-x)^r}{r^r} \sin^2(r\pi) \frac{\Gamma(r)}{1-r}\, dr. \quad \blacksquare \end{aligned}$$

Finally, we investigate the rise of f and its asymptotic behavior in the points $-e$ and 0.

THEOREM 2.4. *Let x be in $(-e, 0)$; then*:

(i) $f(x) \sim (\sqrt{2}/(e^{3/2}\pi))(e+x)^{1/2}$, *as* $x \to -e_+$,

(ii) $f(x) \sim \ln^{-2}(-x)(-x)^{-1}$, *as* $x \to 0_-$,

(iii) f *is increasing strictly on* $(-e, 0)$.

Proof. (i) By Theorem 2.3(ii) and with $u := 1 - \ln(-x)$ we get:

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(s-1)^{s-1}}{\Gamma(s+1)} e^{-s} e^{su}\, ds.$$

Moreover, with Stirling's formula (see [7, p. 294]), as $s \to \infty$, it follows that

$$\frac{(s-1)^{s-1}}{\Gamma(s+1)} e^{-s} = \frac{(s-1)^{s-1}}{s(s-1)\,\Gamma(s-1)} e^{-s}$$
$$\sim \frac{(s-1)^{s-1}\, e^{-s}}{s(s-1)(s-1)^{s-1}\, e^{-(s-1)}} \sqrt{\frac{s-1}{2\pi}} \sim \frac{s^{-3/2}}{e\sqrt{2\pi}}.$$

Thus the conditions of an Abel theorem for inversions of Laplace transforms (Theorem 3 in [4, p. 503]) are fulfilled and as $u \to 0_+$ and $x \to -e_+$ respectively f holds:

$$f(x) \sim \frac{1}{e\sqrt{2\pi}} \frac{u^{1/2}}{\Gamma(\frac{3}{2})} = \frac{\sqrt{2}}{e\pi} (1 - \ln(-x))^{1/2}$$
$$= \frac{\sqrt{2}}{e\pi} \left(-\ln\left(1 - \frac{x+e}{e}\right)\right)^{1/2}$$
$$\sim \frac{\sqrt{2}}{e\pi} \left(\frac{x+e}{e}\right)^{1/2} = \frac{\sqrt{2}}{e^{3/2}\pi} (x+e)^{1/2}.$$

(ii) Using Theorem 2.3(iii), we have $f(x) = \tilde{f}(-\ln(-x))/-x$, with

$$\tilde{f}(-\ln(-x)) := \int_0^\infty e^{-(-\ln(-x))\, r} \frac{\sin^2(\pi r)}{\pi^2 r^r} \frac{\Gamma(r)}{1-r}\, dr.$$

Since

$$\frac{\sin^2(\pi r)}{\pi^2 r^r} \frac{\Gamma(r)}{1-r} = \frac{\sin(\pi r)}{\pi r^r} \frac{1}{\Gamma(2-r)} \sim r, \qquad \text{as} \quad r \to 0_+,$$

we can apply an Abel theorem for Laplace integrals (Theorem 33.3 in [3, p. 241]) and obtain the asymptotic behavior of $\tilde{f}$ as $-\ln(-x) \to \infty$,

$$\tilde{f}(-\ln(-x)) \sim \frac{\Gamma(1+1)}{(-\ln(-x))^2}, \qquad \text{as} \quad -\ln(-x) \to \infty,$$

and that means $f(x) \sim \ln^{-2}(-x)(-x)^{-1}$, as $x \to 0_-$.

(iii) Using Theorem 2.3 (iii) and a theorem on the derivative of Laplace integrals (Theorem 6.1 in [3, p. 37]), it follows that

$$f'(x) = \frac{1}{(-x)^2 \pi^2} \int_0^\infty \frac{(-x)^r}{r^r} \sin^2(\pi r)\, \Gamma(r)\, dr,$$

which is greater than 0 for all x in $(-e, 0)$, and thus the proof is completed. ▮

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